PMM U.S.S.R.,Vol.44,pp.794-800 0021-8928/81/6 0794 $7.50/0

UDC 539.383

PRESSURE OF A CIRCULAR STAMP ON A COMPOSITE LAYER*

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The method of dual integral equations is used to obtain a solution to the problem of a rigid circular stamp pressing on an elastic composite layer, with a cylindrical surface separating the materials. A large number of papers have already been published, dealing with the mechanics of multilayered media in which the surfaces separating the layers from each other do not intersect the outer boundary (see references in /1/). The formulation and methods of solution of the fundamental boundary value problems can be found for such media in the monographs /2,3/.

Considerably less attention has been given to the study of the boundary value problems for composite media in which the surfaces separating the layers do intersect the outer boundary. The authors of /4,5/ call such media the regions with transverse (vertical) layer folding. Out of the publications dealing with the methods of solving contact problems for transversely layered regions, attention should be drawn to /4—11/ (**).

Axisymmetric problems of torsion were studied in /12,13/ et al. For the regions indicated the three-dimensional problems, in particular the axisymmetric problems of the pressure exerted by a stamp applied to a part of the outer boundary, remain practically untouched.

1. A layer resting on a rigid elastic foundation is composed of the domains 1)$(0\leqslant r'\leqslant a, 0\leqslant z'\leqslant H)$ and 2)$(a\leqslant r'<\infty, 0\leqslant z'\leqslant H)$, occupied by materials with different elastic constants ν_1, G_1 and ν_2, G_2. A smooth rigid stamp bounded by a convex surface of revolution is applied to the segment $0\leqslant r'\leqslant b,\ b\geqslant a,\ z'=H$. We assume that there is no friction between the bodies 1 and 2, between the stamp and the layer, and between the layer and the foundation. The force impressing the stamp acts along the axis of the cylindrical inclusion. We assume for simplicity that the boundary of the layer outside the stamp is free of external forces. The boundary conditions of the problem in question have the form

$$w^{(2)}(r,h)=w_0+f(r)\quad(\varepsilon\leqslant r\leqslant 1),\quad \sigma_{zz}^{(2)}(r,h)=0\quad(1<r<\infty) \tag{1.1}$$

$$w^{(1)}(r,h)=w_0+f(r)\quad(0\leqslant r\leqslant\varepsilon),\quad w^{(i)}(r,0)=\tau_{rz}^{(i)}(r,0)=\tau_{rz}^{(i)}(r,h)=\tau_{rz}^{(i)}(\varepsilon,z)=0 \tag{1.2}$$

$$u^{(1)}(\varepsilon,z)=u^{(2)}(\varepsilon,z),\quad \sigma_{rr}^{(1)}(\varepsilon,z)=\sigma_{rr}^{(2)}(\varepsilon,z),\quad r=\frac{r'}{b},\quad z=\frac{z'}{b},\quad \varepsilon=\frac{a}{b},\quad h=\frac{H}{b}$$

where $w_0<0$ denotes the progressive displacement of the stamp along the z-axis, $f(r)$ is the equation of the stamp surface, and $i=1,2$.

We write the solution of the Lamé equations for the domain 1 in the form /14/

$$u^{(1)}=\sum_{j=1}^{\infty}(A_jz\operatorname{sh}\lambda_jz+C_j\operatorname{ch}\lambda_jz)J_1(\lambda_jr)+B_0r+\sum_{n=1}^{\infty}\left[B_n^{(1)}\frac{4(1-\nu_1)}{k_n}I_1(k_nr)-B_n^{(1)}rI_0(k_nr)-D_n^{(1)}I_1(k_nr)\right]\cos k_nz$$

$$w^{(1)}=A_0z+\sum_{j=1}^{\infty}\left(A_j\frac{3-4\nu_1}{\lambda_j}\operatorname{sh}\lambda_jz-A_jz\operatorname{ch}\lambda_jz-C_j\operatorname{sh}\lambda_jz\right)J_0(\lambda_jr)+\sum_{n=1}^{\infty}[B_n^{(1)}rI_1(k_nr)+D_n^{(1)}I_0(k_nr)]\sin k_nz,\quad k_n=\frac{\pi n}{h}$$

and for the domain 2 we use, as the solution of the Lamé equations, the expressions

*Prikl.Matem.Mekhan.,44,No.6,1111-1119,1980

**) See also Babloian A.A. and Gulkanian N.O. Plane contact problem of the theory of elasticity for two truncated wedges. Tezisy dokl. Vses. konf. po smeshannym zadacham mekaniki deformiruemogo tela, ch. 1, Izd-vo Rostov. Univ., 1977; Minasian A.F. and Tonoian V.S. Contact problem for a composite half-plane with a finite vertical cut. Tezisy dokl. Vses. konf. po smeshannym zadacham mekhaniki deformiruemogo tela. ch.2. Izd. Rostov. Univ. 1977.

$$u^{(2)} = \frac{D_0}{r} + \sum_{n=1}^{\infty}\left\{-D_n^{(2)}K_1(k_nr) + B_n^{(2)}\left[rK_0(k_nr) + \frac{4(1-\nu_2)}{k_n}K_1(k_nr)\right]\right\}\cos k_nz -$$

$$\int_0^\infty \{A(\lambda)\operatorname{ch}\lambda z + C(\lambda)[(3-4\nu_2)\operatorname{ch}\lambda z + \lambda z\operatorname{sh}\lambda z]\}J_1(\lambda r)\,d\lambda$$

$$w^{(2)} = -\sum_{n=1}^{\infty}[D_n^{(2)}K_0(k_nr) - B_n^{(2)}rK_1(k_nr)]\sin k_nz + \int_0^\infty [A(\lambda)\operatorname{sh}\lambda z + C(\lambda)\lambda z\operatorname{ch}\lambda z]J_0(\lambda r)\,d\lambda$$

Here $A_0, B_0, D_0, A_j, C_j, B_n^{(1)}, B_n^{(2)}, D_n^{(1)}, D_n^{(2)}$ are arbitrary constants, $A(\lambda), C(\lambda)$ are arbitrary functions, $J_n(x)$ is a Bessel function of the first kind, $I_n(x)$ and $K_n(x)$ are the MacDonald functions, and λ_j are positive roots of the equation $J_1(\lambda\varepsilon) = 0$. Terms containing the constants A_0, B_0 and D_0 correspond to the elementary solutions for a finite cylinder and a layer weakened by a cylindrical cavity. A formal reason for including the term A_0z in the solution is provided by the fact that in this case the system of functions $J_0(\lambda_jr)$ becomes complete only after incorporating a constant into it. When choosing the positive roots of the equation $J_0(\lambda\varepsilon) = 0$ as λ_j, we must put $A_0 = 0$ /11/.

Satisfying the conditions (1.2), eliminating $A(\lambda), D_n^{(i)}$ from the resulting equations and putting $C(\lambda)\operatorname{sh}\lambda h = C^*(\lambda)$, we obtain

$$B_0 = D_0 + \frac{2\nu_2}{h}\int_0^\infty \lambda^{-1}C^*(\lambda)J_1(\lambda\varepsilon)\,d\lambda, \quad \frac{G_1(B_0 + A_0\nu_1)}{1-2\nu_1} = -G_2\left[D_0 + \frac{2\nu_2}{h}\int_0^\infty \lambda^{-1}C^*(\lambda)J_1(\lambda\varepsilon)\,d\lambda\right] \tag{1.3}$$

$$A_0 = \frac{w_0}{h}, \quad A_j\lambda_jh\operatorname{ch}\lambda_jh - (1-2\nu_1)A_j\operatorname{sh}\lambda_jh + C_j\lambda_j\operatorname{sh}\lambda_jh = 0$$

$$\varkappa_1A_j\cdot\operatorname{sh}\lambda_jh = \frac{\lambda_ja_j}{\varepsilon^2J_0^2(\lambda_j\varepsilon)}, \quad a_j = \int_0^\varepsilon rf(r)J_0(\lambda_jr)\,dr$$

$$B_n^{(1)} = \frac{\varkappa_2}{\varkappa_1}\left[\frac{K_1(k_n\varepsilon)}{I_1(k_n\varepsilon)}B_n^{(2)} - \frac{2(-1)^nk_n}{I_1(k_n\varepsilon)}\int_0^\infty \frac{\lambda C^*(\lambda)J_1(\lambda\varepsilon)\,d\lambda}{h(k_n^2+\lambda^2)}\right]$$

$$\frac{(-1)^nB_n^{(2)}}{K_1(k_n\varepsilon)} = \frac{2\varkappa_1k_nI_1^2(k_n\varepsilon)}{h\Delta(k_n\varepsilon)}\left\{-\frac{2G_1}{\varkappa_1}k_n^2\sum_{j=1}^{\infty}\frac{\lambda_ja_j}{J_0(\lambda_j\varepsilon)(k_n^2+\lambda_j^2)^2} + \right.$$

$$\int_0^\infty\left[\frac{G_1\varkappa_2}{\varkappa_1}\frac{\Delta_1(k_n\varepsilon)}{I_1^2(k_n\varepsilon)}\frac{\lambda}{k_n^2+\lambda^2} + 2G_2\varkappa_2\frac{\lambda}{k_n^2+\lambda^2} + \right.$$

$$\left.\left. 2G_2\frac{\lambda^3k_n\varepsilon K_0(k_n\varepsilon)}{(k_n^2+\lambda^2)^2K_1(k_n\varepsilon)}\right]J_1(\lambda\varepsilon)C^*(\lambda)\,d\lambda - 2G_2\times\int_0^\infty\frac{\lambda^2k_n^2\varepsilon}{(k_n^2+\lambda^2)^2}J_0(\lambda\varepsilon)C^*(\lambda)\,d\lambda\right\}$$

$$\Delta(k_n\varepsilon) = G_1\varkappa_2K_1^2(k_n\varepsilon)\Delta_1(k_n\varepsilon) - G_2\varkappa_1I_1^2(k_n\varepsilon)\Delta_2(k_n\varepsilon)$$

$$\Delta_1(k_n\varepsilon) = k_n^2\varepsilon^2[I_0^2(k_n\varepsilon) - I_1^2(k_n\varepsilon)] - 2\varkappa_1I_1^2(k_n\varepsilon)$$

$$\Delta_2(k_n\varepsilon) = k_n^2\varepsilon^2[K_0^2(k_n\varepsilon) - K_1^2(k_n\varepsilon)] - 2\varkappa_2K_1^2(k_n\varepsilon), \quad \varkappa_i = 1-\nu_i \quad (i = 1, 2)$$

Realizing the mixed conditions (1.1), we obtain the following dual integral equations:

$$\int_0^\infty C^*(\lambda)J_0(\lambda r)\,d\lambda = g(r) \quad (\varepsilon \leqslant r \leqslant 1), \quad \int_0^\infty \frac{C^*(\lambda)}{1-G(\lambda h)}\lambda J_0(\lambda r)\,d\lambda = F(r) \quad (1 < r < \infty) \tag{1.4}$$

$$G(\lambda h) = \frac{\lambda h + \operatorname{sh}\lambda h e^{-\lambda h}}{\lambda h + \operatorname{sh}\lambda h\operatorname{ch}\lambda h}, \quad g(r) = -\frac{w_0 + f(r)}{2\varkappa_2}, \quad F(r) = -\sum_{n=1}^{\infty}\{P_nK_0(k_nr) + (-1)^nB_n^{(2)}[2K_0(k_nr) - k_nrK_1(k_nr)]\}$$

$$P_n = \frac{k_n}{K_1(k_n\varepsilon)}\left[(-1)^nB_n^{(2)}\varepsilon K_0(k_n\varepsilon) + \frac{4}{h}\int_0^\infty\frac{\lambda^3C^*(\lambda)J_1(\lambda\varepsilon)\,d\lambda}{(k_n^2+\lambda^2)^2}\right]$$

Let us extend, in accordance with /3/, the right-hand part of the first equation of (1.4) to the segment $0 \leqslant r \leqslant 1$, and put $C^*(\lambda) = [1 - G(\lambda h)]\psi(\lambda)$. Then we have

$$\int_0^\infty \psi(\lambda)J_0(\lambda r)\,d\lambda = \int_0^\infty G(\lambda h)\psi(\lambda)J_0(\lambda r)\,d\lambda + g(r) \quad (0 \leqslant r \leqslant 1), \quad \int_0^\infty \lambda\psi(\lambda)J_0(\lambda r)\,d\lambda = F(r) \quad (1 < r < \infty) \tag{1.5}$$

Following /7/ we mutliply the first equation of (1.5) by $r(t^2-r^2)^{-1/2}\,dr$, integrate with respect to r from zero to t and differentiate the resulting expression with respect to t. Next we multiply the second equation of (1.5) by $r(r^2-t^2)^{-1/2}\,dr$ and integrate with respect to r from t to ∞. Using now the equations

$$\frac{d}{dt}\int_0^t \frac{rJ_0(\lambda r)\,dr}{\sqrt{t^2-r^2}} = \cos\lambda t, \quad \int_t^\infty \frac{rJ_0(\lambda r)\,dr}{\sqrt{r^2-t^2}} = \frac{\cos\lambda t}{\lambda}$$

and the Fourier cosine transform we obtain, after some manipulations,

$$\psi(\lambda) = \frac{1}{\pi}\int_0^\infty G(vh)\left[\frac{\sin(\lambda+u)}{\lambda+u} + \frac{\sin(\lambda-u)}{\lambda-u}\right]\psi(u)\,du + \sum_{n=1}^\infty \frac{e^{-k_n}}{k_n(k_n^2+\lambda^2)}\left[P_nF_1(\lambda,k_n) + (-1)^n B_n^{(2)}F_2(\lambda,k_n)\right] + r(\lambda) \tag{1.6}$$

$$F_1(\lambda,k_n) = \lambda\sin\lambda - k_n\cos\lambda, \quad F_2(\lambda,k_n) = \left(\frac{2\lambda^2}{k_n^2+\lambda^2} - k_n\right)F_1(\lambda,k_n) - \lambda\sin\lambda$$

$$r(\lambda) = \frac{2}{\pi}\int_0^1 \cos\lambda t\left[\frac{d}{dt}\int_0^t \frac{rg(r)\,dr}{\sqrt{t^2-r^2}}\right]dt$$

Substituting into (1.6) the expressions for P_n and $B_n^{(2)}$, we arrive at a regular integral Fredholm equation of the second kind, i.e. at an integral equation of the second kind with a continuous, square summable kernel and a free term.

2. Let us consider in more detail the problem of impressing a stamp with a plane foundation. In this case we have

$$C_j = A_j = a_j = 0 \quad (j = 1, 2, \ldots), \quad r(\lambda) = -\frac{\gamma_0}{\pi\varkappa_2}\frac{\sin\lambda}{\lambda}$$

Let us put

$$(-1)^n B_n^{(1)} = \frac{x_n^{(1)}}{k_nI_1(k_n\varepsilon)}, \quad (-1)^n B_n^{(2)} = \frac{x_n^{(2)}}{k_nK_1(k_n\varepsilon)} \tag{2.1}$$

Then by virtue of (1.3) we obtain

$$\varkappa_1 x_n^{(1)} = \varkappa_2 x_n^{(2)} - \frac{2}{h}\varkappa_2 k_n^2 \int_0^\infty \frac{\lambda[1-G(\lambda h)]\psi(\lambda)}{k_n^2+\lambda^2}J_1(\lambda\varepsilon)\,d\lambda \tag{2.2}$$

$$x_n^{(2)} = \frac{2\varkappa_1 k_n^2 I_1^2(k_n\varepsilon)K_1^2(k_n\varepsilon)}{h\Delta(k_n\varepsilon)}\int_0^\infty \left\{\left[\frac{G_1\varkappa_2}{\varkappa_1}\frac{\Delta_1(k_n\varepsilon)}{I_1^2(k_n\varepsilon)} + 2G_2\varkappa_2 + 2G_2\frac{k_n\varepsilon K_0(k_n\varepsilon)\lambda^2}{K_1(k_n\varepsilon)(k_n^2+\lambda^2)}\right]J_1(\lambda\varepsilon) - 2G_2\frac{k_n^2\lambda\varepsilon}{k_n^2+\lambda^2}J_0(\lambda\varepsilon)\right\}\times \frac{\lambda[1-G(\lambda h)]}{k_n^2+\lambda^2}\psi(\lambda)\,d\lambda$$

Taking into account the substitutions (2.1), we arrive at the following expressions for the contact stresses under the stamp:

$$\frac{\sigma_{zz}^{(1)}(r,h)}{2G_1} = \frac{2\nu_1B_0 + (1-\nu_1)A_0}{1-2\nu_1} + \sum_{n=1}^\infty \frac{x_n^{(1)}}{k_n}S_1(k_n,r) \tag{2.3}$$

$$\frac{\sigma_{zz}^{(2)}(r,h)}{2G_2} = -\int_0^\infty \lambda\psi(\lambda)J_0(\lambda r)\,d\lambda + \sum_{n=1}^\infty \frac{x_n^{(2)}}{k_n}S_2(k_n,r) - \frac{4}{h}\sum_{n=1}^\infty \frac{k_nK_0(k_nr)}{K_1(k_n\varepsilon)}\int_0^\infty \frac{\lambda^3[1-G(\lambda h)]\psi(\lambda)}{(k_n^2+\lambda^2)^2}J_1(\lambda\varepsilon)\,d\lambda \tag{2.4}$$

$$S_1(k_n,r) = \frac{k_n[rI_1(k_nr)I_1(k_n\varepsilon) - \varepsilon I_0(k_nr)I_2(k_n\varepsilon)]}{I_1^2(k_n\varepsilon)}$$

$$S_2(k_n,r) = \frac{k_n[rK_1(k_nr)K_1(k_n\varepsilon) - \varepsilon K_0(k_nr)K_2(k_n\varepsilon)]}{K_1^2(k_n\varepsilon)}$$

After some manipulations the expression (2.4) can be written in the form

$$\frac{\sigma_{zz}^{(2)}(r,h)}{2G_2} = \frac{\gamma}{\sqrt{t^2-r^2}} - \frac{2}{\pi}\int_r^1 \frac{dt}{\sqrt{t^2-r^2}}\int_0^\infty uG(uh)\psi(u)\sin ut\,du - \tag{2.5}$$

$$\int_r^1 \frac{dt}{\sqrt{t^2-r^2}} \sum_{n=1}^{\infty} \left[P_n + \frac{x_n^{(2)}}{k_n K_1(k_n \varepsilon)} (2 - k_n t) \right] e^{-k_n t}$$

$$\gamma = \frac{w_0}{\pi \varkappa_2} - \frac{2}{\pi} \int_0^{\infty} G(uh) \psi(u) \cos u \, du - \sum_{n=1}^{\infty} \left[P_n + \frac{x_n^{(2)}}{k_n K_1(k_n \varepsilon)} (1 - k_n) \right] \frac{e^{-k_n}}{k_n}$$

Both integrals in the first equation of (2.5) vanish as $r \to 1$, consequently we have

$$\sigma_{zz}^{(2)}(r,h) \sim \frac{2G_2\gamma}{\sqrt{1-r^2}} \quad (r \to 1)$$

It is clear that the difficulties encountered in investigating the convergence of the series appearing in the expressions for $\sigma_{zz}^{(1)}(r,h), \sigma_{zz}^{(2)}(r,h)$ arise only in the near vicinity of the line $r = \varepsilon$ and on the line $r = \varepsilon$ itself. Using the integral equation for the function sought, we can confirm that

$$\int_0^{\infty} \frac{\lambda [1 - G(\lambda h)] \psi(\lambda)}{k_n^2 + \lambda^2} J_1(\lambda \varepsilon) \, d\lambda = \frac{\pi}{2} k_n^{-1} e^{-k_n} I_1(k_n \varepsilon) \Big\{ - \frac{w_0}{\pi \varkappa_2} -$$

$$\frac{2}{\pi} k_n \int_0^{\infty} \frac{u G(uh) \psi(u) \sin u}{k_n^2 + u^2} du + \frac{2}{\pi} k_n^2 \int_0^{\infty} \frac{G(uh) \psi(u) \cos u}{k_n^2 + u^2} du +$$

$$k_n \sum_{m=1}^{\infty} \frac{e^{-k_m} P_m}{k_m (k_m + k_n)} - k_n \sum_{m=1}^{\infty} \frac{e^{-k_m} [k_m (k_m + k_n) - k_n] \, x_m^{(2)}}{k_m^2 K_1(k_m \varepsilon)(k_m + k_n)^2} \Big\}$$

hence

$$\int_0^{\infty} \frac{\lambda [1 - G(\lambda h)] \psi(\lambda)}{k_n^2 + \lambda^2} J_2(\lambda \varepsilon) \, d\lambda \sim - \frac{\pi}{2} \gamma k_n^{-1} e^{-k_n} I_1(k_n \varepsilon) \quad (n \to \infty)$$

The validity of the following asymptotic equations is proved in the same manner:

$$\int_0^{\infty} \frac{\lambda^3 [1 - G(\lambda h)] \psi(\lambda)}{(k_n^2 + \lambda^2)^2} J_1(\lambda \varepsilon) \, d\lambda \sim \frac{\pi}{4} \gamma e^{-k_n} [I_1(a_n \varepsilon) - \varepsilon I_0(k_n \varepsilon)]$$

$$\int_0^{\infty} \frac{\lambda^2 [1 - G(\lambda h)] \psi(\lambda)}{(k_n^2 + \lambda^2)^2} J_0(\lambda \varepsilon) \, d\lambda \sim \frac{\pi}{4} \gamma k_n^{-1} e^{-k_n} [e I_1(k_n \varepsilon) - I_0(k_n \varepsilon)]$$

Separating in (2.2) the principal parts of the integrals, we find that

$$\frac{x_n^{(1)}}{\varkappa_2}, \ \frac{x_n^{(2)}}{\varkappa_1} \sim \frac{\pi G_2 \gamma (1 - \varepsilon)}{h (G_1 \varkappa_2 + G_2 \varkappa_1)} k_n^2 e^{-k_n} [I_0(k_n \varepsilon) + I_1(k_n \varepsilon)] \quad (n \to \infty)$$

Consequently the convergence of every series in (2.3) and (2.4) is characterized by an exponentially decaying general term, and this implies the boundedness of the contact stresses under the stamp near the line $r = \varepsilon$ as well as on the line $r = \varepsilon$. The latter holds also for the stresses $\sigma_{zz}^{(1)}(r, 0)$, $\sigma_{zz}^{(2)}(r, 0)$.

Assuming $G_1 = 0$, $\nu_1 = 0$, which corresponds to the contact problem for a layer weakened by a cylindrical cavity we obtain, as before, the following regular integral Fredholm equation of the second kind

$$\psi(\lambda) = \int_0^{\infty} K(\lambda, u) \psi(u) \, du - \frac{w_0}{\pi \varkappa_2} \frac{\sin \lambda}{\lambda} \tag{2.6}$$

$$K(\lambda, u) = \frac{1}{\pi} G(uh) \Big[\frac{\sin(\lambda + u)}{\lambda + u} + \frac{\sin(\lambda - u)}{\lambda - u} \Big] - \frac{4}{h} \sum_{n=1}^{\infty} \frac{[1 - G(uh)] \, e^{-k_n}}{\Delta_2(k_n \varepsilon)(k_n^2 + \lambda^2)} [F_1(\lambda, k_n) H_1(u, k_n) + F_2(\lambda, k_n) H_2(u, k_n)]$$

$$H_1(u, k_n) = \Big[\frac{u^3 (k_n^2 \varepsilon^2 + 2\varkappa_2) K_1(k_n \varepsilon)}{(k_n^2 + u^2)^2} + \frac{\varkappa_2 k_n \varepsilon u K_0(k_n \varepsilon)}{k_n^2 + u^2} \Big] J_1(u \varepsilon) - \frac{u^2 k_n^3 \varepsilon^2 K_1(k_n \varepsilon)}{(k_n^2 + u^2)^2} J_0(u \varepsilon)$$

$$H_2(u, k_n) = \Big[\frac{\varkappa_2 u K_1(k_n \varepsilon)}{k_n^2 + u^2} + \frac{u^3 k_n \varepsilon K_0(k_n \varepsilon)}{(k_n^2 + u^2)^2} \Big] J_1(u \varepsilon) - \frac{u^2 k_n^2 \varepsilon K_1(k_n \varepsilon)}{(k_n^2 + u^2)^2} I_0(u \varepsilon)$$

We note that $F(r) \to 0$ as $a \to 0$ $(\varepsilon \to 0)$ and equation (2.6) is transformed to the integral equation of the contact problem for a homogeneous dense layer

$$\psi(\lambda) = \frac{1}{\pi}\int_0^\infty G(uh)\left[\frac{\sin(\lambda+u)}{\lambda+u} + \frac{\sin(\lambda-u)}{\lambda-u}\right]\psi(u)\,du - \frac{w_0}{\pi\varkappa_2}\frac{\sin\lambda}{\lambda} \tag{2.7}$$

The case $b \to \infty$ $(b \gg a)$ also corresponds to (2.7).

The process of solving the equations obtained is accompanied by the usual computational problem arising at small values of $h = H/b$ /15/, and this necessitates additional, not at all trivial investigations. In the present case a more detailed analysis of the properties of the solutions is needed also in the case when h are sufficiently large.

As an example we shall consider the integral equation (2.6). The arguments that follow will also refer to the case of the integral equation of the initial problem. It can be shown that the computational problems arising in the process of solving equations (2.6) and (2.7) are analogous, at small h, to those of the integral equation of the first kind appearing in the contact problem for a layer /15/. A method of solving equations of the type (2.7) for large h is given in /16/.

Let us denote by $K_1(\lambda, u)$ and $K_2(\lambda, u)$ the first and second terms of the kernel $K(\lambda, u)$ of the equation (2.6). Let also $C[0, \infty)$ be a space of continuous functions bounded on the semiaxis, with the norm

$$\| y(\lambda) \| = \sup_{(\lambda)} | y(\lambda) | \quad (0 \leqslant \lambda < \infty)$$

Lemma. The function

$$y(\lambda) = \int_0^\infty K(\lambda, u)\, x(u)\,du$$

belongs to the space $C[0, \infty)$ for any $x(\lambda) \in L_2(0, \infty)$ and $h > 0$.

The validity of the lemma follows from the Cauchy—Buniakowski inequality, continuity and boundedness of the function $K(\lambda, u)$, and the estimates

$$\left|\frac{e^{-k_n}}{k_n^2+\lambda^2} F_1(\lambda, k_n)\right| = \left|\int_1^\infty e^{-k_n t}\cos\lambda t\,dt\right| \leqslant k_n^{-1} e^{-k_n} \tag{2.8}$$

$$\left|\frac{e^{-k_n}}{k_n^2+\lambda^2} F_2(\lambda, k_n)\right| = \left|\int_1^\infty (k_n t - 1)\, e^{-k_n t}\cos\lambda t\,dt\right| < k_n^{-1}(k_n + 2)\, e^{-k_n}$$

$$|K_1(\lambda, u)| \leqslant \frac{2}{\pi} G(uh) \quad (0 \leqslant \lambda, u < \infty), \quad |K_2(\lambda, u)| < d_1 u^{-1} |J_1(u\varepsilon)| + d_2 u^{-2} |J_0(u\varepsilon)|$$

$$(0 \leqslant \lambda < \infty,\ u \geqslant u_0 > 0,\ d_i > 0,\ d_i = \text{const})$$

Since the free term of (2.6) belongs to the space $C[0, \infty)$, the lemma implies that only the functions belonging to the space $C[0, \infty)$ can be solutions of the equation (2.6).

Theorem 1. Operator Kx defined at $h > 0$ by the equation

$$Kx = \int_0^\infty K(\lambda, u)\, x(u)\,du$$

is completely continuous in $C[0, \infty)$. The validity of Theorem 1 can be established using the lemma, the estimates (2.8) and the Arzel theorem.

Theorem 2. Integral equation (2.6) has a unique solution belonging to the space $C[0, \infty)$ at any $h > 0$. The proof of Theorem 2, with the lemma and Theorem 1 taken into account, is analogous to that given in /16/.

Carrying out the substitution $w^{(i)} \to -w^{(i)}$, $z \to h - z$ in the initial expressions for the components of the displacement vector and using the relation /14/

$$\lim_{\alpha\to\infty} \sum_{n=1}^\infty \frac{1}{\alpha} f\left(\frac{n}{\alpha}\right) = \int_0^\infty f(s)\,ds$$

we pass, as $h \to \infty$, from the equation (2.6) to the integral equation

$$\psi_\infty(\lambda) = \int_0^\infty k(\lambda, u)\,\psi_\infty(u)\,du + \frac{w_0}{\pi\varkappa_2}\frac{\sin\lambda}{\lambda} \quad (w_0 > 0) \tag{2.9}$$

$$k(\lambda, u) = \frac{4}{\pi}\int_0^\infty \frac{e^{-s}}{\Delta_2(s\varepsilon)(s^2+\lambda^2)}\,[F_1(\lambda, s)\, H_1(u, s) + F_2(\lambda, s)\, H_2(u, s)]\,ds$$

The functions $F_i(\lambda, s)$, $H_i(u, s)$, $\Delta_2(s\varepsilon)$ are obtained by carrying out the substitution $k_n \to S$ in the expressions for $F_i(\lambda, k_n)$, $H_i(u, k_n)$, $\Delta_2(k_n\varepsilon)$ given above. Equation (2.9) is also obtained in the course of solving the initial problem for a half-space with a cylindrical cavity.

Using the estimates analogous to (2.8) and the inequalities

$$\left|\frac{e^{-s}}{s^2+\lambda^2} F_1(\lambda, s)\right| < \frac{2e^{-s}}{\lambda}, \quad \left|\frac{e^{-s}}{s^2+\lambda^2} F_2(\lambda, s)\right| < \frac{2(s+2)}{\lambda} e^{-s} \quad (s > 0, \lambda \geqslant \lambda_0 > 0)$$

we can show that the kernel $k(\lambda, u)$ is square integrable and continuous function in the domain $0 \leqslant \lambda, u < \infty$. From this it follows that equation (2.9) is a regular, integral Fredholm equation of the second kind, and the assertions analogous to those given above, hold for this equation.

Let us now denote by $\psi_h(\lambda)$ the solution of (2.6) corresponding to a specified value of the parameter h.

Theorem 3. $\psi_h(\lambda) \to \psi_\infty(\lambda)$ is uniform in λ as $h \to \infty$.

The validity of the theorem follows from the complete continuity of the operator Kx, completeness of the space $C[0,\infty)$ and uniqueness of the solutions of the integral equations (2.6) and (2.9).

The assertions given here do not exhaust the problem of the structure of the solutions of the integral equations. This is true, in particular, with regard to the feasibility of obtaining a practically manageable analytic estimate of the rate of convergence of $\psi_h(\lambda)$ to $\psi_\infty(\lambda)$, depending on h. We know /17,18/ that a problem of this type is complicated even in the case when the integral equation is defined on a finite interval and its kernel can be expressed explicitly. Notwithstanding that, the problems connected with the possibility of replacing a solution of the problem for a composite layer (a layer with a cavity) with a solution for a composite half-space (half-space with a cavity), or the problem of selecting the latter solution as the initial approximation for sufficiently large h, can be solved numerically. Naturally, the known difficulties of summing the series at sufficiently large h encountered here must be overcome.

We shall indicate another circumstance related to the correctness of the contact problem for a composite half-space. When the conditions $\tau_{rz}^{(1)}(\varepsilon, z) = 0$ hold and stresses are absent at infinity, the solution of the problem in question will be correct if the stresses $\sigma_{zz}^{(1)}(r, 0)$ are self-equilibrated. Indeed, in this case we have

$$\sigma_{zz}^{(1)}(r, z) = 2G_1 \lim_{h\to\infty} \sum_{n=1}^{\infty} (-1)^n k_n^{-1} S_1(k_n r) \cos k_n z \to 0 \quad (z \to \infty), \quad \iint_{r\leqslant\varepsilon} \sigma_{zz}^{(1)}(r, z)\, r\, dr\, d\theta = 0 \quad (0 \leqslant z < \infty)$$

The author thanks V.S. Protsenko for showing unceasing interest in this work.

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Translated by L.K.